The existence of constrained Willmore surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$

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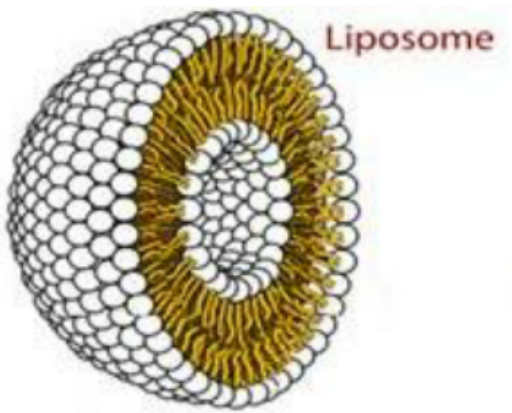
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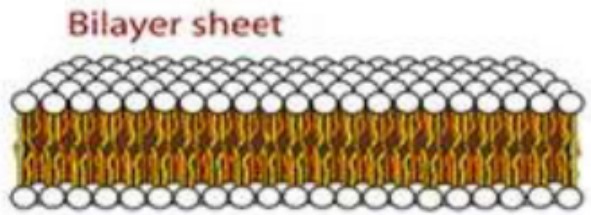
Willmore Energy

- Let $f : M \to \mathbb{R}^3$ or $\mathbb{R}^4$ be an immersion of a surface M. Its Willmore energy is

$$\mathcal{W}(f) = \int_M H^2 \, dA = \frac{1}{4} \int_M (k_1 - k_2)^2 + 2\pi\chi(M).$$

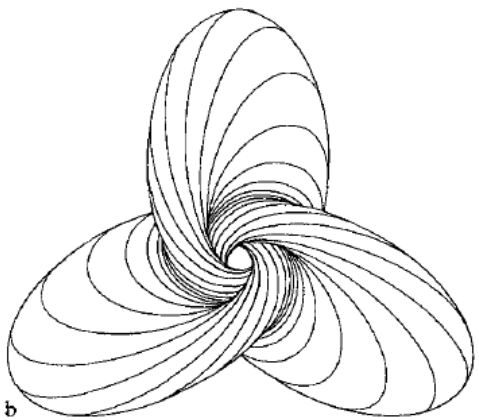
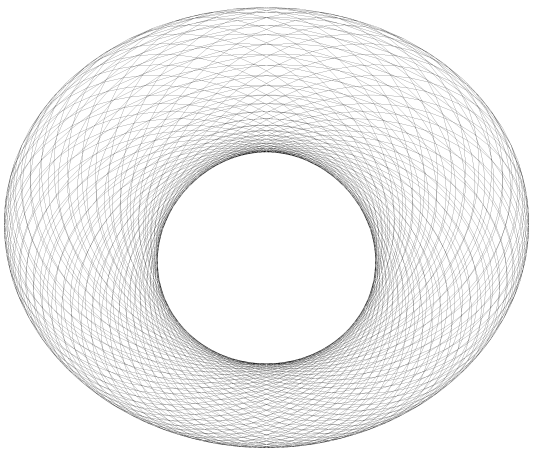
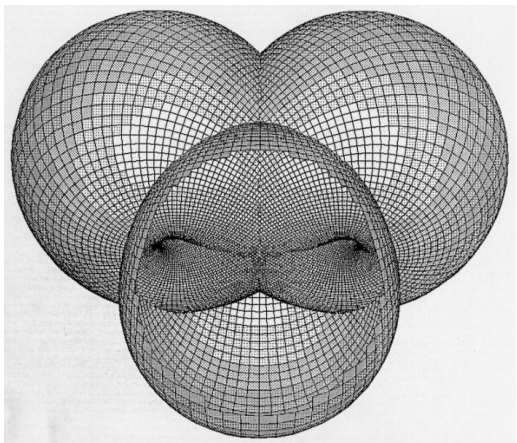
- Introduced by Willmore [Wil65], he suggests that the infinum might be an interesting topological invariant.
- A Willmore surface is a critical point of this functional.
- The Willmore energy is a conformal invariant.
- A constrained Willmore surface is a critical point of the Willmore energy for a fixed conformal class of surface M (Riemann surface).





Relation to other Surfaces

- Minimal surfaces are Willmore surfaces.
- But there are others, eg Hopf tori [Pin85].
- All surfaces with constant mean curvature are constrained Willmore surfaces.

Left: Hopf torus (Pinkall). Centre: Homogeneous tori in $\mathbb{S}^3$. Right: Cut-away of Wente torus (Sterling). Also see GeometrieWerkstatt Gallery from N. Schmitt.

Willmore Conjecture and Major Results

- In Willmore's paper, he conjectured that for tori $\mathcal{W}(f) \geq 2\pi^2$. Checked this for tori of revolution, minimum for radii 1 and $\sqrt{2}$.
- Very useful result by Li and Yau [LY82]: If a map is k-to-1 at some point then $\mathcal{W} \geq 4\pi k$.
- Bryant found all Willmore spheres [Bry84]. Described them as conformally equivalent to a special family of minimal surfaces, and then reduced this further to a problem in algebraic geometry. They all have Willmore energy a multiple of 4π.
- Existence of tori minimisers proved by Simon [Sim93].
- Willmore conjecture proved by Marques and Neves [MN14].
- Papers of Bauer, Kuwert, Rivière, Schätzle succeeded in proving the existence of constrained Willmore surfaces [Riv14].

Theorem

Let X be a compact Riemann surface. The Willmore functional attains a minimum on the space of smooth branched conformal maps $F : X \to \mathbb{R}^4$. The Willmore functional also attains a minimum on the space of smooth branched conformal maps $F : X \to \mathbb{R}^3$ with only even-order roots of dF. Furthermore, if the Willmore functional of this minimizer in $\mathbb{R}^3$ is smaller than 12π, then it is an immersion. Finally, in both cases, if the Willmore functional of the minimizer is smaller than 8π, then the minimizer is an immersed embedding.

Conformal immersions into $\mathbb{H}$

- Riemann surface X, conformal immersion $F : X \to \mathbb{H}$.
  The conformal structure on X is given by Hodge star. $\mathbb{H}$ has length and angle.
- The immersion is conformal if there exists $N \in \mathbb{S}^2$ such that

$$- * dF = N\, dF.$$

- A function $f : X \to \mathbb{C}$ is holomorphic iff

$$\tfrac{1}{2}(d - i * d)f = d\bar{z}\tfrac{\partial}{\partial \bar{z}} f = 0.$$

- An immersion $F : X \to \mathbb{H}$ is conformal with left normal N iff

$$\tfrac{1}{2}(d - N * d)F = 0.$$

Kodaira map

- Consider a Riemann surface X and a holomorphic map to $\mathbb{C}P^n$.
  In components $f = [f_0 : \cdots : f_n]$ for meromorphic functions f_k on X.
- Example: consider the double periodic functions 1, $\wp$, $\wp'$. These are meromorphic functions on the torus $\mathbb{C}/\Gamma$.
  Embedding of torus $z \mapsto [1 : \wp(z) : \wp'(z)]$ in $\mathbb{C}P^2$.
  Image is the algebraic curve $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.
- Meromorphic functions are holomorphic sections of line bundles.
  Changing by a cocycle doesn't affect the embedding:
  $[1 : \wp(z) : \wp'(z)] = [z^3 : z^3\wp(z) : z^3\wp'(z)]$.

Complex Quaternionic Line Bundles

- Theory of Pedit-Pinkall [PP98], et al.
- We take N to be a (left) complex structure on the trivial (right) $\mathbb{H}$-line bundle $X \times \mathbb{H}$. Complex $\mathbb{H}$-vector spaces split into $E \otimes_{\mathbb{C}} \mathbb{H}$ for E a $\mathbb{C}$-vector space, $E = \{x \mid Nx = xi\}$
- We also have the trivial connection d on $X \times \mathbb{H}$, which splits $d''_N = \frac{1}{2}(d - N * d)$.
  The commuting part $\bar{\partial}_E := (d''_N)^+ = \frac{1}{2}(d''_N - Nd''_N N)$ makes E into a holomorphic $\mathbb{C}$-line bundle.
  The anticommuting part is tensorial, a *potential*, called the Hopf field.
- Sections in the kernel of $d''_N = \bar{\partial}_E - V$ are called V-holomorphic.
- We have two V-holomorphic sections.
  The trivial section 1 is V-holomorphic: $d1 = 0$ so $d''_N 1 = 0$.
  The conformal map F considered is V-holomorphic:
  $d''_N F = \frac{1}{2}(d - N * d)F = 0$ is exactly the condition to be conformal.

Quaternionic Kodaira Representation

On a Riemann surface X to every smooth map $F : X \to \mathbb{H}$ we can associate a quadruple (E, V, υ, ϕ), consisting of

- a holomorphic $\mathbb{C}$-line bundle E,
- a smooth potential V and
- two linear-independent V-holomorphic sections υ and ϕ, such that υ has no roots,

with $F = \upsilon^{-1}\phi$.
The left normal of F obeys $N = \upsilon^{-1}\mathrm{i}\upsilon$.
The line bundle E is uniquely determined up to isomorphism class. Fixing E, the associated quadruples have the form $(E, gVg^{-1}, g\upsilon, g\phi)$ for any holomorphic function $g : X \to \mathbb{C}^\times$.

Quaternionic Kodaira Representation, Weak Version

Definition (Admissible map)

$F \in W^{1,1}_{\mathrm{loc}}(X, \mathbb{H})$, dF almost everywhere rank 2, conformal with $N \in W^{1,2}_{\mathrm{loc}}(X, \mathbb{H})$.

Theorem

On a Riemann surface X to every admissible map $F : X \to \mathbb{H}$ we can associate a quadruple (E, V, υ, ϕ), consisting of

- *a holomorphic $\mathbb{C}$-line bundle E,*
- *an L^2 potential V and*
- *two linear-independent V-holomorphic sections υ and ϕ, such that υ has no roots,*

with $F = \upsilon^{-1}\phi$. The left normal of F obeys $N = \upsilon^{-1}\mathrm{i}\upsilon$. Same uniqueness conditions for E.

Example of Admissible Map

- $X = \mathbb{C}P^1$,

$$V = \begin{cases} d\bar{z}\mathrm{j}V_0 & |z| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the sections are

$$v, \phi = \begin{cases} \text{Bessel functions}(|z|)e^{i\theta} & |z| < 1 \\ \text{holomorphic} & \text{otherwise,} \end{cases}$$

- Let's look at the demo!

Weierstrass representation

$$dF = -\upsilon^{-1} d\upsilon \upsilon^{-1} \phi + \upsilon^{-1} d\phi = \upsilon^{-1}(d\phi - d\upsilon\upsilon^{-1}\phi)$$
$$= \upsilon^{-1} dz(\partial - \partial\upsilon\upsilon^{-1})\phi = \overline{(\mathrm{j}\upsilon)}^{-1} \mathrm{j} dz(\partial - \partial\upsilon\upsilon^{-1})\phi.$$

- Define $\partial\upsilon\upsilon^{-1} = -(B + U)$.
- $\psi = (\partial - \partial\upsilon\upsilon^{-1})\phi$ is U-holomorphic and
- $\chi = \overline{(\mathrm{j}\upsilon)}^{-1}$ is $\mathrm{k}\bar{U}\mathrm{k}$-holomorphic.
- $dF = \bar{\chi}\mathrm{j} dz\psi$.
- Roots of ψ are the branch points of F.

Weierstrass representation in $\mathbb{R}^3$

- We use the imaginary quaternions $\operatorname{Im}\mathbb{H}$ to describe $\mathbb{R}^3$.
- Consider immersions with $\overline{dF} = -dF \Leftrightarrow dF \in \operatorname{Im}\mathbb{H}$.
- At any point we ask "Given a value of χ, what are the possible values of ψ?" The trick is to write ψ as a multiple of the non-zero χ:

$$\begin{aligned}\left\{\psi \in \mathbb{H} \mid \overline{\bar{\chi}\mathrm{j}dz\psi} = -\bar{\chi}\mathrm{j}dz\psi\right\} &= \left\{\psi = g\chi \mid g \in \mathbb{H},\, \bar{g}\mathrm{j}dz = \mathrm{j}dzg\right\} \\ &= \left\{\psi = g\chi \mid g \in \mathbb{C}\right\}.\end{aligned}$$

- F is a map into $\operatorname{Im}\mathbb{H}$ if there exists a holomorphic section g of KE^2 with $\psi = g\chi$ and $U = g\mathrm{k}\bar{U}\mathrm{k}g^{-1}$.
- If F is unbranched (or more generally, the roots of ψ are even order), then $\chi = \psi$.

Associated Family of Minimal Surfaces

- Cute observation: if $U = 0$, then χ, ψ are 0-holomorphic.
- In particular, $h\psi$ is again holomorphic for any $h : X \to \mathbb{C}$. Doesn't change the normal.
- Take $h \in \mathbb{S}^1$. Then the Weierstrass representation is no longer exact. For the catenoid:

$$\int_{|z|=1} dF_h = \int_0^{2\pi} \overline{(1 + e^{-\mathrm{i}\theta}\mathrm{k})}\,\mathrm{j}\,d\theta \mathrm{i} e^{\mathrm{i}\theta} h(1 + e^{-\mathrm{i}\theta}\mathrm{k}) = -4\pi\mathrm{i}\,\mathrm{Im}\, h.$$

This gives the catenoid-helicoid family.

Willmore Energy and Potentials

- $H = \frac{1}{2}\left[4\,\mathrm{Re}(\mathrm{j}\alpha_1 U)\Xi^{\alpha_1} + 4\,\mathrm{Re}(\mathrm{j}\alpha_2 U)\Xi^{\alpha_2}\right] = 2\chi^{-1}\,\mathrm{j}U\,\bar{\psi}^{-1}.$
- $\mathcal{W}(F) = \int_X |H|^2 |dF|^2 = 4\int_X |U|^2\,|dz|^2 = 4\|U\|_2^2.$
- For compact X we also get $\mathcal{W}(F) = 4\pi\,\deg(E) + 4\|V\|_2^2.$

Limits of Admissible Maps

- Let X be a compact Riemann surface.
- We control the limit of an admissible map by controlling the limit of the Kodaira or Weierstrass representation.
- Consider a sequence with bounded degree of the line bundles and bounded L^2 norm of the potentials (bounded Willmore energy).
- The limit of a potential (really the mean curvature density) can develop point measures.
- Around a point with concentrating energy, take a disc and embed it into $\mathbb{P}^1$. Use Möbius transformations to deconcentrate the energy.
- Use techniques from the resolvent of Dirac operator on $\mathbb{P}^1$ to analyze the operators $\bar{\partial} - U$ on $\mathbb{P}^1$.
- Theorem: A sequence of U_n-holomorphic sections can develop poles in the limit, but the pole order is bounded by the concentration of Willmore energy divided 4π.

- We use the quaternionic Plucker formula [FLPP01], which relates the Willmore energy to the degree of E.
- We control the limit of the sections on a well-behaved part of X and extend by unique continuation.

Theorem

Let $n \in \{3, 4\}$, let X be a compact Riemann surface and $(F_n)_{n\in\mathbb{N}}$ a sequence of admissible maps from X to $\mathbb{R}^n$ with bounded Willmore functionals. Then the compositions of a subsequence of $(F_n)_{n\in\mathbb{N}}$ with a sequence of conformal transformations converge to an admissible map F.

Example of Limit Admissible Map

- $X = \mathbb{C}P^1$,

$$V = \begin{cases} d\bar{z}\mathrm{j}R^{-1}V_0 & |z| < R \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\|V\|_2^2 = \int_{B(0,R)} R^{-2}|V_0|^2 \, d\mu = R^{-2}|V_0|^2 \pi R^2 = \pi |V_0|^2$$

Regularity of Constrained Willmore

- $\mathcal{W}(F) = 4\|U\|_2^2 = 4\pi \deg(E) + 4\|V\|_2^2$.
- Constrained Willmore means that the derivative of $\mathcal{W}$ vanishes on the tangent space of $\mathrm{Kod}(X, E)$.
- Since $\delta\mathcal{W} = 8\langle \delta V, V\rangle$, we are constrained Willmore
  iff δV is orthogonal to V for all tangent vectors $(\delta V, \delta\upsilon, \delta\phi)$ to $\mathrm{Kod}(X, E)$
  iff there exist $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic sections υ^*, ϕ^* with

$$V = 2\mathrm{k}\upsilon^*\overline{\upsilon} + 2\mathrm{k}\phi^*\overline{\phi}.$$

- This is an analog of Euler-Lagrange equation.
- Elliptic bootstrapping gives smoothness.

Integrability and CMC

- Flat connection $d - (dv)v^{-1} = (\partial + B + U) + (\bar{\partial} - V)$.
- If there exists a $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic section v^* such that $\overline{B'} + V = 2\mathrm{k}v^*\overline{v}$ for some B' then there is a family of flat connections

$$(\partial + B - B'(1-\lambda) + U) + (\bar{\partial} - V\lambda).$$

- If $B' = 0$, then you have CMC.
-

Holomorphic:	$V = 0$
CMC:	$V = 2\mathrm{k}v^*\overline{v}$
C Willmore:	$V = 2\mathrm{k}v^*\overline{v} + 2\mathrm{k}\phi^*\overline{\phi}$
???:	$V = 2\mathrm{k}v^*\overline{v} + 2\mathrm{k}\phi^*\overline{\phi} + \ldots.$

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