Kodaira And Weierstrass Representations of (Weakly) Conformal Maps

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Overview

Part I: Kodaira

- Conformal maps into $\mathbb{R}^4$
- Kodaira representation
- Space of conformal immersions

Part II: Weierstrass

- Weierstrass representation
- Differential geometry
- Minimal, CMC, constrained Willmore

Kodaira map

- Consider a Riemann surface X and a holomorphic map to $\mathbb{C}P^n$.
  In components $f = [f_0 : \cdots : f_n]$ for meromorphic functions f_k on X.
- Example: consider the double periodic functions 1, $\wp$, $\wp'$. These are meromorphic functions on the torus $\mathbb{C}/\Gamma$.
  Embedding of torus $z \mapsto [1 : \wp(z) : \wp'(z)]$ in $\mathbb{C}P^2$.
  Image is the algebraic curve $Y_0 Y_2^2 = 4 Y_1^3 - g_2 Y_0^2 Y_1 - g_3 Y_0^3$.
- Meromorphic functions are holomorphic sections of line bundles.
  Changing by a cocycle doesn't affect the embedding:
  $[1 : \wp(z) : \wp'(z)] = [z^3 : z^3\wp(z) : z^3\wp'(z)]$.

Quaternions

- $\mathbb{H} = \mathbb{R}\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k}\rangle$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = \mathrm{k}$, $\mathrm{ij} = -\mathrm{ji}$.
- $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$, $\mathbb{S}^3 = \{x \in \mathbb{H} \mid |x|^2 = 1\}$,
  $\mathbb{S}^2 = \mathbb{S}^3 \cap \operatorname{Im}\mathbb{H} = \{x \in \mathbb{H} \mid x^2 = -1\}$.
- We identify an oriented plane in $\mathbb{H}$ with the rotation that fixes the plane and negates the perpendicular plane.
  Such rotations are $x \mapsto Nx(-R)$ for $N, R \in \mathbb{S}^2$. $x \mapsto Nx$ commutes with this rotation, and $N^2 = -1$, so is a quarter turn. We choose the sign of N to match the orientation of the plane.
- By construction, the oriented plane is $\{x \mid Nx = xR\}$.
- N, R are called the *left and right normals* of the plane.

Conformal immersions

- Riemann surface X, immersion $F : X \to \mathbb{H}$. Conformal means angle-preserving. The conformal structure on X is given by Hodge star. $\mathbb{H}$ has length and angle.
- The immersion is conformal if there exists $N \in \mathbb{S}^2$ such that

$$- * dF = N\, dF.$$

- Admissible map: $F \in W^{1,1}_{\mathrm{loc}}(X, \mathbb{H})$, dF a.e. rank 2, weakly conformal, $N \in W^{1,2}_{\mathrm{loc}}(X, \mathbb{H})$.

Catenoid

- $F : \mathbb{C}^\times \to \mathbb{H}$

$$\begin{aligned} F(z) &= \ln(z\bar{z})\mathrm{i} + (\bar{z} + z^{-1})\mathrm{j} \\ &= 2u\mathrm{i} + (e^u + e^{-u})e^{iv}\mathrm{j}, && \text{for } z = exp(u + iv). \end{aligned}$$

$$dF = (1 - z^{-1}\mathrm{k})\mathrm{j}\, dz\, (1 + z^{-1}\mathrm{k}), \qquad \text{for } dz = dx + idy.$$

$$\begin{aligned} -*\, dF &= (1 - z^{-1}\mathrm{k})\mathrm{j}\, (\mathrm{i}dz)\, (1 + z^{-1}\mathrm{k}) \\ &= (1 - z^{-1}\mathrm{k})(-\mathrm{i})\mathrm{j}\, dz\, (1 + z^{-1}\mathrm{k}) \\ &= N(1 - z^{-1}\mathrm{k})\mathrm{j}\, dz\, (1 + z^{-1}\mathrm{k}), \\ \text{where } N &= (1 - z^{-1}\mathrm{k})(-\mathrm{i})(1 - z^{-1}\mathrm{k})^{-1}. \end{aligned}$$

Complex Quaternionic Line Bundles

- A function $f : X \to \mathbb{C}$ is holomorphic iff

$$\tfrac{1}{2}(d - i * d)f = d\bar{z}\frac{\partial}{\partial \bar{z}}f = 0.$$

- An immersion $F : X \to \mathbb{H}$ is conformal with left normal N iff

$$\tfrac{1}{2}(d - N * d)F = 0.$$

- We take N to be a (left) complex structure on the trivial (right) $\mathbb{H}$-line bundle $X \times \mathbb{H}$. Complex $\mathbb{H}$-vector spaces split into $E \otimes_{\mathbb{C}} \mathbb{H}$ for E a $\mathbb{C}$-vector space, $E = \{x \mid Nx = xi\}$
- We also have the trivial connection d on $X \times \mathbb{H}$, which splits $d''_N = \frac{1}{2}(d - N * d)$.
  The commuting part $\bar{\partial}_E := (d''_N)^+ = \frac{1}{2}(d''_N - Nd''_N N)$ makes E into a holomorphic $\mathbb{C}$-line bundle.
  The anticommuting part is tensorial, a *potential*, called the Hopf field.

Catenoid

$$N = (1 - z^{-1}\mathrm{k})(-\mathrm{i})(1 - z^{-1}\mathrm{k})^{-1}.$$
$$E = \{(z, (1 - z^{-1}\mathrm{k})\mathrm{j}h) \in X \times \mathbb{H} \mid h \in \mathbb{C}\}$$
$$E \otimes_{\mathbb{C}} \mathbb{H} = \{(z, (1 - z^{-1}\mathrm{k})\mathrm{j}h) \in X \times \mathbb{H} \mid h \in \mathbb{H}\} = X \times \mathbb{H}$$

Choose the non-vanishing section $\mathbf{e} = (1 - z^{-1}\mathrm{k})\mathrm{j}$ of E. It is holomorphic:

$$d\mathbf{e} = \mathbf{e}\Big[- dz\, z^{-1} - d\bar{z}\mathrm{k}z^{-1}\bar{z}\Big](1 + |z|^2)^{-1} =: \mathbf{e}\Big[dz\,(B + U) - d\bar{z}\,V\Big]$$

We also have two sections in the kernel of d''_N, namely $\mathbf{1}$ and F. In the trivialization given by $\mathbf{e}$, write $\mathbf{1} = \mathbf{e}\upsilon$ and $F = \mathbf{e}\upsilon F = \mathbf{e}\phi$, we have

$$(\bar{\partial} - V)\upsilon = 0, \qquad (\bar{\partial} - V)\phi = 0.$$

Sections of $E \otimes_{\mathbb{C}} \mathbb{H}$ that obey this equation are called V-holomorphic.

Theorem (Quaternionic Kodaira Representation)

On a Riemann surface X to every admissible map $F : X \to \mathbb{H}$ we can associate a quadruple (E, V, υ, ϕ), consisting of

- *a holomorphic $\mathbb{C}$-line bundle E,*
- *a potential V and*
- *two linear-independent V-holomorphic sections υ and ϕ, such that υ has no roots,*

with $F = \upsilon^{-1}\phi$. The left normal of F obeys $N = \upsilon^{-1}\mathrm{i}\upsilon$.
The line bundle E is uniquely determined up to isomorphism class. Fixing E, the associated quadruples have the form $(E, gVg^{-1}, g\upsilon, g\phi)$ for any holomorphic function $g : X \to \mathbb{C}^\times$.

Möbius transformations

- One of the nice features of the Kodaira representation is that it easily handles Möbius transformations.
- Consider $\tilde{\phi} = \phi\alpha + \upsilon\beta$, $\tilde{\upsilon} = \phi\gamma + \upsilon\delta$ for $\alpha, \beta, \gamma, \delta \in \mathbb{H}$. These new sections are also V-holomorphic.
- The associated map is

$$\begin{aligned} \tilde{F} = \tilde{\upsilon}^{-1}\tilde{\phi} &= (\phi\gamma + \upsilon\delta)^{-1}(\phi\alpha + \upsilon\beta) \\ &= (\upsilon^{-1}\phi\gamma + \delta)^{-1}(\upsilon^{-1}\phi\alpha + \beta) = (F\gamma + \delta)^{-1}(F\alpha + \beta). \end{aligned}$$

Compactified Inverted Catenoid

- Möbius transform the catenoid with $\alpha = \delta = \mathrm{i}$ and $\beta = \gamma = 1$.

$$\tilde{F}(z) = [F + \mathrm{i}]^{-1}\,[F\mathrm{i} + 1] = [zF + z\mathrm{i}]^{-1}\,[zF\mathrm{i} + z]$$

and

$$\lim_{z\to 0} zF(z) = \lim_{z\to 0} z\,\mathrm{ln}(z\bar{z})\mathrm{i} + (z\bar{z} + 1)\mathrm{j} = \mathrm{j}.$$

- Likewise $\tilde{v}, \tilde{\phi}$ extend continuously to $z = 0$.

$$\tilde{v} = v(F + \mathrm{i}) = -\mathrm{j}(1 - z^{-1}\mathrm{k})^{-1}(F + \mathrm{i}) = -\mathrm{j}(z - \mathrm{k})^{-1}(zF + z\mathrm{i}),$$
$$\tilde{\phi} = v(F\mathrm{i} + 1) = -\mathrm{j}(1 - z^{-1}\mathrm{k})^{-1}(F\mathrm{i} + 1) = -\mathrm{j}(z - \mathrm{k})^{-1}(zF\mathrm{i} + z).$$

- The potential V does not extend continuously

$$V = \mathrm{k}z^{-1}\bar{z}(1 + |z|^2)^{-1}.$$

Space of Admissible Maps

- $\Theta(V, v, \phi) = ((\bar{\partial}_E - V)v, (\bar{\partial}_E - V)\phi)$
- For a compact Riemann surface X and a fixed holomorphic $\mathbb{C}$-line bundle E, let $\mathrm{Kod}(X, E)$ denote the subspace of triples

$$\Big\{(V, v, \phi) \in \mathrm{Pot}_E \times H^0(X, W_E^{1,p}) \times H^0(X, W_E^{1,p}) \mid \Theta(V, v, \phi) = 0, v \text{ has no roots}\Big\}.$$

- To apply the implicit function theorem and show that $\mathrm{Kod}(X, E)$ is a submanifold at (V, v, ϕ), need $\ker \Theta' = 0$.

$$\Theta'(\delta V, \delta v, \delta\phi) = ((\bar{\partial}_E - V)\delta v, (\bar{\partial}_E - V)\delta\phi) - (\delta V v, \delta V \phi) =: \mathrm{A} + \mathrm{B}$$

Some linear algebra

- Let $\mathrm{A} : X \to Z$, $\mathrm{B} : Y \to Z$, and $\Theta' : X \times Y \to Z$ by $\Theta'(x, y) = \mathrm{A}(x) + \mathrm{B}(y)$.
- Θ' is surjective if $\tilde{\mathrm{B}} : Y \to Z \to Z/\operatorname{img} \mathrm{A}$ is surjective.
- $\tilde{\mathrm{B}}$ is surjective if $\tilde{\mathrm{B}}^* : (Z/\operatorname{img} \mathrm{A})^* \to Z^* \to Y^*$ is injective.
- $(Z/\operatorname{img} \mathrm{A})^* \cong \ker \mathrm{A}^*$.

Now apply this

- By integration by parts

$$\int_X \bar{\xi} \mathrm{j} dz \wedge d\bar{z}(\bar{\partial} - V)v = \overline{\int_X \bar{v} \mathrm{j} dz \wedge d\bar{z}(\bar{\partial} - \mathrm{k}\bar{V}\mathrm{k})\xi}$$

- $\ker \mathrm{A}^*$ is pairs (v^*, ϕ^*) of $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic sections.

- Taking the real part of the pairing of v^* and $\delta V v$ gives

$$\begin{aligned}\mathrm{Re}\int_X \overline{v^*}\mathrm{j}dz\wedge d\bar{z}\delta V v &= \mathrm{Re}\int_X v\overline{v^*}(-2\mathrm{k}d\mu)\delta V\\ &= \mathrm{Re}\int_X \overline{2\mathrm{k}v^*\overline{v}}\,\delta V\, d\mu = \langle 2\mathrm{k}v^*\overline{v},\delta V\rangle.\end{aligned}$$

- The dual of B is $(v^*,\phi^*)\mapsto 2\mathrm{k}v^*\overline{v}+2\mathrm{k}\phi^*\overline{\phi}$.
- $\tilde{\mathrm{B}}^*$ is not injective if there exist a pair (v^*,ϕ^*) of $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic sections with $v^*\overline{v}+\phi^*\overline{\phi}=0$.
- Rearranging

$$(\phi^*)^{-1}v^* = -\bar{\phi}\bar{v}^{-1} = -\overline{v^{-1}\phi} = -\bar{F}.$$

- Such a pair determines the Christoffel dual of F via $dG=\bar{v}\mathrm{j}dz\phi^*$.

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Weierstrass Representation

- Recall the Kodaira representation $F = \upsilon^{-1}\phi$ for V-holomorphic sections υ, ϕ of $E \otimes_{\mathbb{C}} \mathbb{H}$.
- In fact υ, being non-vanishing, defines a flat connection ∇.
  Let $d\upsilon = -dz(B + U)\upsilon + d\bar{z}V\upsilon$.
  Then $\nabla = d + dz(B + U) - d\bar{z}V$

$$\begin{aligned} 0 = \nabla^2\xi &= \nabla(d\xi + dz(B + U)\xi - d\bar{z}V\xi) \\ &= -dz \wedge d\bar{z}(\bar{\partial} - U)(\partial + B + U)\xi \\ &\quad + dz \wedge d\bar{z}(\partial + B + V)(\bar{\partial} - V)\xi \end{aligned}$$

- Differentiating F gives

$$\begin{aligned} dF &= -\upsilon^{-1}d\upsilon\upsilon^{-1}\phi + \upsilon^{-1}d\phi = \upsilon^{-1}(d\phi - d\upsilon\upsilon^{-1}\phi) \\ &= \upsilon^{-1}dz(\partial\phi - \partial\upsilon\upsilon^{-1}\phi) = \upsilon^{-1}dz(\partial + B + U)\phi \end{aligned}$$

Hence $\psi := (\partial + B + U)\phi$ is U-holomorphic.

- Can we understand υ^{-1} to be holomorphic somehow? Let $\chi = (\overline{\mathrm{j}\upsilon})^{-1}$. Then

$$\begin{aligned} d(\overline{\mathrm{j}\upsilon})^{-1} &= -\mathrm{j}\overline{d\upsilon\,\upsilon^{-1}}\,\bar{\upsilon}^{-1} = -\mathrm{j}\,\overline{(d\bar{z}V - dz(B+U))}\bar{\upsilon}^{-1} \\ &= \mathrm{j}(d\bar{z}(\bar{B}+V) - dzU)\bar{\upsilon}^{-1} \\ &= (dz(B - \mathrm{j}V\mathrm{j}) - d\bar{z}(-\mathrm{j}U\mathrm{j}))\chi. \end{aligned}$$

  This shows that χ is $\mathrm{j}U\mathrm{j}$-holomorphic. Because of general considerations, we actually write this as $\mathrm{k}\bar{U}\mathrm{k}$-holomorphic.
- Hence $dF = \bar{\chi}\mathrm{j}dz\psi$. We call this is the Weierstrass representation of F.

Catenoid

- Recall

$$\upsilon = -\mathrm{j}(1 - z^{-1}\mathrm{k})^{-1}$$
$$\phi = \upsilon F = -\mathrm{j}(1 - z^{-1}\mathrm{k})^{-1}\left[\ln(z\bar{z})\mathrm{i} + (\bar{z} + z^{-1})\mathrm{j}\right]$$
$$B = -z^{-1}(1 + |z|^2)^{-1}, \qquad U = 0, \qquad V = \mathrm{k}z^{-1}\bar{z}(1 + |z|^2)^{-1}.$$

- Thus we have the Weierstrass representation

$$dF = (1 - z^{-1}\mathrm{k})\mathrm{j}\, dz\, (1 + z^{-1}\mathrm{k}) = \upsilon^{-1} dz\, \psi$$
$$\chi = 1 + z^{-1}\mathrm{k}$$
$$\psi = 1 + z^{-1}\mathrm{k}.$$

Note the χ, ψ are indeed 0-holomorphic.

Theorem (Quaternionic Weierstrass Representation)

For any admissible map $F : X \to \mathbb{H}$ on a Riemann surface X there exist

- *a holomorphic $\mathbb{C}$-line bundle E,*
- *a potential $U \in Pot_{KE}$,*
- *a $\mathrm{k}\bar{U}\mathrm{k}$-holomorphic section χ of $E^{-1} \otimes_{\mathbb{C}} \mathbb{H}$ and*
- *a U-holomorphic section ψ of $KE \otimes_{\mathbb{C}} \mathbb{H}$*

with $dF = \bar{\chi}\mathrm{j}dz\psi$.
There is a unique choice of E up to isomorphism such that χ has no roots.
For any E in this isomorphism class the possible quadruples are $(E, gUg^{-1}, g^{-1}\chi, g\psi)$ for a holomorphic function $g : X \to \mathbb{C}^{\times}$.
Vice versa, for any such quadruple (E, U, χ, ψ) any anti-derivative F of $dF = \bar{\chi}\mathrm{j}dz\psi$ defines an admissible map on the universal covering of X.

Left and Right Normals

- Why is this call the Weierstrass representation? From this form we can read off the left and right normals:

$$\begin{aligned} N\,dF &= -*\,dF = \bar{\chi}\mathrm{j}(\mathrm{i}dz)\psi = \bar{\chi}(-\mathrm{i})\bar{\chi}^{-1}\bar{\chi}\mathrm{j}dz\psi \\ &\Rightarrow N = \bar{\chi}(-\mathrm{i})\bar{\chi}^{-1} \quad \Rightarrow N = -\chi^{-1}\mathrm{i}\chi,, \\ dF\,R &= -*\,dF = \bar{\chi}\mathrm{j}(\mathrm{i}dz)\psi = \bar{\chi}\mathrm{j}dz\psi\psi^{-1}\mathrm{i}\psi \\ &\Rightarrow R = \psi^{-1}\mathrm{i}\psi. \end{aligned}$$

  The sections parameterize the normals.

- Weierstrass representation of $-\bar{F}$ is

$$-\overline{dF} = -\bar{\psi}d\bar{z}(-\mathrm{j})\chi = \bar{\psi}\mathrm{j}dz\chi.$$

  Normals $N_{-\bar{F}} = -\psi^{-1}\mathrm{i}\psi = -R_F$ and $R_{-\bar{F}} = \chi^{-1}\mathrm{i}\chi = -N_F$.

Christoffel Dual

- Two admissible maps $F, \tilde{F} : X \to \mathbb{H}$ are Christoffel duals if left and right normals of $\tilde{F}$ are the negatives of the left and right normals of F respectively.
- $\tilde{N} = -N = -\upsilon^{-1}\mathrm{i}\upsilon$, hence we have a Weierstrass section $\tilde{\chi} = \upsilon$.
- It has a Weierstrass representation $d\tilde{F} = \bar{\upsilon}\mathrm{j}dz\phi^*$ for some $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic section ϕ^*.
- $(\phi^*)^{-1}\mathrm{i}\phi^* = \tilde{R} = -R = N_{-\bar{F}}$
- $-\bar{F}$ has a Kodaira representation $-\bar{F} = (\phi^*)^{-1}\upsilon^*$.
- Rearranging $(\phi^*)^{-1}\upsilon^* = -\bar{F} = -\overline{\upsilon^{-1}\phi}$ gives $\upsilon^*\overline{\upsilon} + \phi^*\overline{\phi} = 0$, an element of the kernel from the use of the inverse function theorem applied to $\mathrm{Kod}(X, E)$.

Conformal maps into $\mathbb{R}^3$

- Consider immersions with $\overline{dF} = -dF \Leftrightarrow dF \in \operatorname{Im}\mathbb{H}$.
- At any point we ask "Given χ, what are the possible values of ψ?" The key is to write ψ as a multiple of the non-zero χ:

$$\begin{aligned}\left\{\psi \in \mathbb{H} \mid \overline{\bar{\chi}\mathrm{j}dz\psi} = -\bar{\chi}\mathrm{j}dz\psi\right\} &= \left\{\psi = g\chi \mid g \in \mathbb{H},\, \bar{g}\mathrm{j}dz = \mathrm{j}dzg\right\} \\ &= \left\{\psi = g\chi \mid g \in \mathbb{C}\right\}.\end{aligned}$$

- If ψ is U-holomorphic, this implies $\bar{\partial}g = 0$ and $g\mathrm{k}\bar{U}\mathrm{k} = Ug$.
- The roots of g are the roots of ψ. If all roots are even order $g = f^2$.

$$\begin{aligned}&\tilde{\chi} = f\chi, \quad \tilde{\psi} = f^{-1}\psi, \quad \tilde{U} = f^{-1}Uf. \\ \Rightarrow \quad &\tilde{\psi} = f^{-1}\psi = f^{-1}g\chi = f\chi = \tilde{\chi}.\end{aligned}$$

- $R = \psi^{-1}\mathrm{i}\psi = \chi^{-1}g^{-1}\mathrm{i}g\chi = \overline{\bar{\chi}(-\mathrm{i})\bar{\chi}^{-1}} = \bar{N} = -N$.

Second Fundamental Form

- The left and right normals describe the tangent plane $\{\bar{\chi}\mathrm{j}\alpha\psi \mid \alpha \in \mathbb{C}\}$ and the perpendicular plane $\{\bar{\chi}\alpha\psi \mid \alpha \in \mathbb{C}\}$.
- A perpendicular vector field $\Xi^\alpha : z \mapsto \bar{\chi}(z)\alpha(z)\psi(z)$.
- The Weingarten operator $A^\alpha = -\frac{1}{2}\Big[d(\bar{\chi}\alpha\psi) - N d(\bar{\chi}\alpha\psi)R\Big] = \frac{1}{2}\Big[\overline{dN}\,\Xi^{\mathrm{i}\alpha} + \Xi^{\mathrm{i}\alpha}\,dR\Big]$.
- The second fundamental form $\mathrm{I\!I}^\alpha(X,Y) = \langle dF(X), A^\alpha(Y)\rangle = |\chi|^2|\psi|^2 \operatorname{Re}\Big[d\bar{z}\otimes dz\, 2\operatorname{Re}(\mathrm{j}\alpha U) + d\bar{z}\otimes d\bar{z}(\mathrm{j}\alpha V' - V\mathrm{j}\alpha)\Big]$.
- $H = \dfrac{1}{2}\Big[4\operatorname{Re}(\mathrm{j}\alpha_1 U)\Xi^{\alpha_1} + 4\operatorname{Re}(\mathrm{j}\alpha_2 U)\Xi^{\alpha_2}\Big] = 2\chi^{-1}\,\mathrm{j}U\,\bar{\psi}^{-1}$.

Associated Family of Minimal Surfaces

- Cute observation: if $U = 0$, then χ, ψ are 0-holomorphic.
- In particular, $h\psi$ is again holomorphic for any $h : X \to \mathbb{C}$. Doesn't change the normal.
- Take $h \in \mathbb{S}^1$. Then the Weierstrass representation is no longer exact. For the catenoid:

$$\int_{|z|=1} dF_h = \int_0^{2\pi} (\overline{1 + e^{-\mathrm{i}\theta}\mathrm{k}})\,\mathrm{j}\,d\theta \mathrm{i} e^{\mathrm{i}\theta} h(1 + e^{-\mathrm{i}\theta}\mathrm{k}) = -4\pi\mathrm{i}\,\mathrm{Im}\, h.$$

This gives the catenoid-helicoid family.

Constrained Willmore

- $\mathcal{W}(F) = \int_X |H|^2 |dF|^2 = 4 \int_X |U|^2 \, |dz|^2 = 4\|U\|_2^2$.
- For compact X we also get $\mathcal{W}(F) = 4\pi \deg(E) + 4\|V\|_2^2$.
- Constrained Willmore means that the derivative of $\mathcal{W}$ vanishes on the tangent space of $\mathrm{Kod}(X, E)$.
- Since $\delta\mathcal{W} = 8\langle \delta V, V \rangle$, we are constrained Willmore iff all tangent vectors $(\delta V, \delta\upsilon, \delta\phi)$ to $\mathrm{Kod}(X, E)$ have that δV is orthogonal to V.

- It's again time for horrible linear algebra!
- $\Theta(V, \upsilon, \phi) = ((\bar\partial_E - V)\upsilon, (\bar\partial_E - V)\phi)$
  $\Theta'(\delta V, \delta\upsilon, \delta\phi) = ((\bar\partial_E - V)\delta\upsilon, (\bar\partial_E - V)\delta\phi) - (\delta V\upsilon, \delta V\phi) =: \mathrm{A} + \mathrm{B}$
  Tangent space of $\mathrm{Kod}(X, E)$ is the kernel of Θ'.
- $A^* : (\upsilon^*, \phi^*) \mapsto ((\bar\partial - V)\upsilon^*, (\bar\partial - V)\phi^*)$,
  $B^* : (\upsilon^*, \phi^*) \mapsto 2\mathrm{k}\upsilon^*\overline{\upsilon} + 2\mathrm{k}\phi^*\overline{\phi}$.
- $(\delta V, \delta\upsilon, \delta\phi)$ is tangent
  $\Leftrightarrow \mathrm{B}(\delta V) \in \operatorname{img} \mathrm{A}$
  $\Leftrightarrow \mathrm{B}(\delta V) \in (\ker \mathrm{A}^*)^\perp$
- The real part of the pairing of υ^* and $\delta V\upsilon$ gives $\langle 2\mathrm{k}\upsilon^*\overline{\upsilon}, \delta V\rangle$.
- $\mathrm{B}(\delta V) \in (\ker \mathrm{A}^*)^\perp \Leftrightarrow \langle \mathrm{B}^*(\ker \mathrm{A}^*), \delta V\rangle = 0$.
- Therefore $\delta\mathcal{W} = 8\langle \delta V, V\rangle = 0$ iff there exist $\mathrm{k}\bar V\mathrm{k}$-holomorphic sections υ^*, ϕ^* with

$$V = 2\mathrm{k}\upsilon^*\overline{\upsilon} + 2\mathrm{k}\phi^*\overline{\phi}.$$

Regularity

- If we have constrained Willmore, the holomorphicity equations become

$$
\begin{aligned}
0 &= (\bar\partial - V)v = \bar\partial v - 2\mathrm{k}(v^*\overline{v} + \phi^*\overline{\phi})v \\
0 &= (\bar\partial - V)\phi = \bar\partial \phi - 2\mathrm{k}(v^*\overline{v} + \phi^*\overline{\phi})\phi \\
0 &= (\bar\partial - \mathrm{k}\bar{V}\mathrm{k})v^* = \bar\partial v^* - 2\mathrm{k}(v\overline{v^*} + \phi\overline{\phi^*})v^* \\
0 &= (\bar\partial - \mathrm{k}\bar{V}\mathrm{k})\phi^* = \bar\partial \phi^* - 2\mathrm{k}(v\overline{v^*} + \phi\overline{\phi^*})\phi^*.
\end{aligned}
$$

- This is a non-linear elliptic system. By a paper of Morrey (1958), the solutions are analytic.

Integrability and CMC

- Recall the flat connection $(\partial + B + U) + (\bar{\partial} - V)$ defined by υ.
- If there exists a $\mathrm{k}\bar{V}\mathrm{k}$-holomorphic section υ^* such that $\overline{B'} + V = 2\mathrm{k}\upsilon^*\overline{\upsilon}$ for some B' then there is a family of flat connections

$$(\partial + B - B'(1-\lambda) + U) + (\bar{\partial} - V\lambda).$$

- If $B' = 0$, then you have CMC.
-

Holomorphic:	$V = 0$
CMC:	$V = 2\mathrm{k}\upsilon^*\overline{\upsilon}$
C Willmore:	$V = 2\mathrm{k}\upsilon^*\overline{\upsilon} + 2\mathrm{k}\phi^*\overline{\phi}$
???:	$V = 2\mathrm{k}\upsilon^*\overline{\upsilon} + 2\mathrm{k}\phi^*\overline{\phi} + \ldots.$